Control and Instrumentation 3 Laboratory Report

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Section 2: Static Characterisation of DC Motors

Exercise 2.3.1: Calculate and Plot DC Motor Gain

Before we can plot the gain of the DC motor vs. input voltage we need to perform the Section 2 experiment from the Lab Manual [1]. We select our first disc ($57g$) and click on the "Voltage (open loop)" setting. We increase the voltage from $0V$ to $10V$ starting in steps of 0.1, then 0.2 and then 0.5. We use smaller step increments at lower input voltages. At each voltage step we use the "Data Snapshot" tool to record the angular velocity (ω) of the disc. When we're done we save the data as a CSV. Now we run the experiment again this time going from $0V$ to $-10V$ with the same voltage steps. We then repeat for a second disc ($110g$). Now using the formula below which is given in the lab manual:

$$\omega = K_v \times V_{in} \tag{2.1}$$

we can calculate the gain of our motor (the motor constant K_v) at each input voltage. We then plot our gain against input voltage for each set of results. The four graphs are pictured in figure 1. We can see our motor constant (K_v) settles at $\simeq 47$.

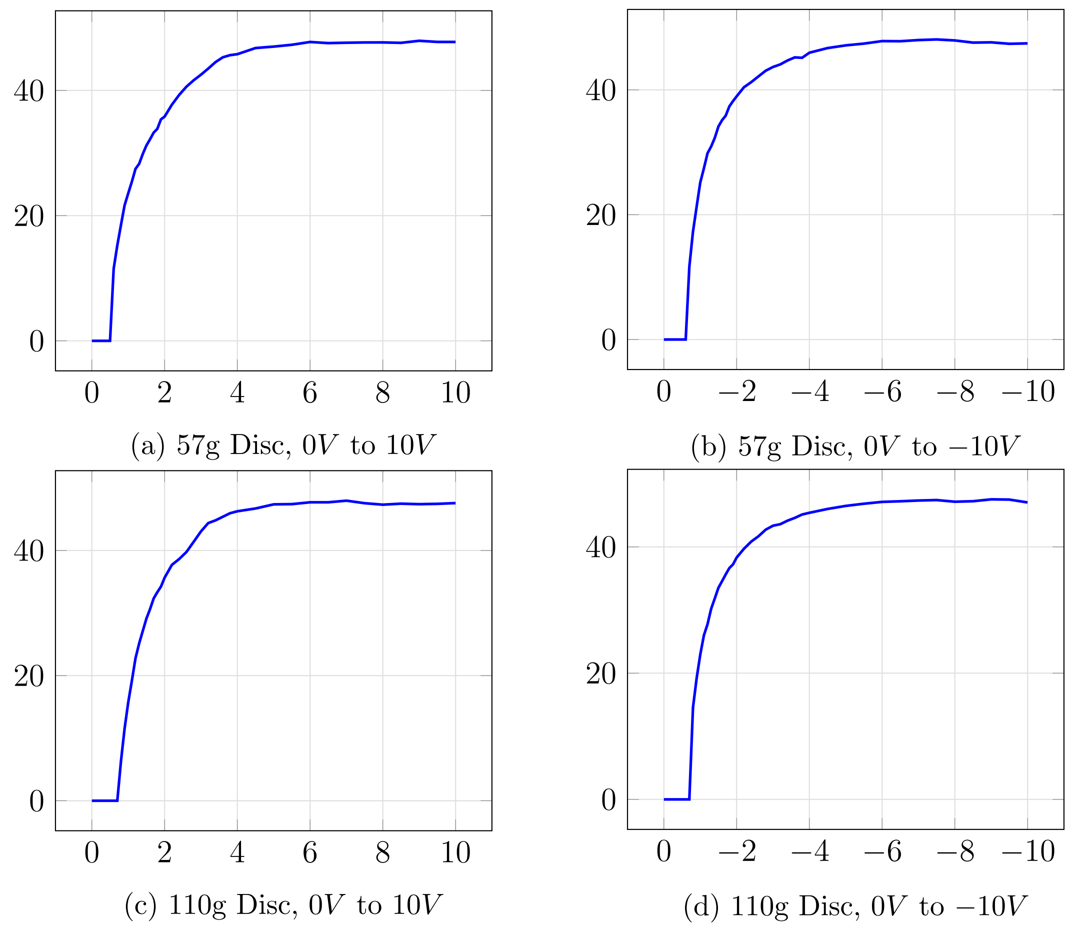


Figure 1: Exercise 2.3.1, Gain vs V_{in}

We notice that the graphs do not follow the simplified DC motor equation given in the lab manual (our value of K_v isn't constant). The angular velocity remains at 0 until a certain "threshold" voltage is reached. Angular velocity, and therefore gain then starts to increase until the graph flattens out at a final value and starts to obey the DC motor equation 2.1 given. We also see slight differences between the $57g$ and $110g$ plots.

Exercise 2.3.2: Reasons for Non-Perfect Match of Response

The non-perfect match of the DC motor equation could come from the fact that equation 2.1 will only be true if our motor torque remains constant according to the full equation: $\omega = \frac{V}{k} - \frac{T}{k^2}R$. At low voltages, the torque of the motor is also low. The inertia of our disc load and other mechanical factors such as air resistance prevent the disc from spinning with angular velocity (ω). As we increase our voltage, current increases and torque increases according to the relationship: $\tau = k_t I_m$ until it is high enough to partially overcome the inertia of the load at which point the disc starts to move.

The torque then continues increasing slightly until it reaches a **constant final value** where it is high enough to fully overcome the load. At this point the relationship between input velocity and ω becomes constant which is when our motor constant K_v stabilises. Internal DC motor losses could also cause non-linearities. This non-perfect match would be problematic in real world applications. For example, Surveillance Radar systems rely on very constant and predictable angular velocity which could be hard to achieve with a varying motor constant. This means that systems could only be operated above certain voltage thresholds where a linear relationship between V_{in} and ω could be expected. This would limit the range of operation of the system.

Exercise 2.3.3 (a): Calculate Max. Pulse Count Measured

Given our encoder has a resolution of 1000CPR and a sampling period of $0.1s$ we can calculate the maximum pulse count by using the maximum speed reached in the experiment. The highest speed reached was 477rad/s by the $57g$ disc. We can use the formula:

$$C_p = \omega r T_s \tag{2.2}$$

where C_p is pulse count, ω is angular velocity, r is resolution and T_s is sampling time. We convert ω to rev/s: $\frac{477}{2\pi} = 75.92$ rev/s, and then sub our numbers in:

$$C_p = \omega r T_s = (75.92)(1000)(0.1) = 7592$$

Exercise 2.3.3 (b): Calculate Max. ω the System can Measure

Given that the pulse count is stored in a 15 bit word we know the maximum pulse count (C_p) is: $2^{15} = 32,768$. Now using equation 2.2 again, this time rearranged:

$$\omega = \frac{C_p}{rT_s} = \frac{(32768)}{(1000)(0.1)} = 327.68$$

This is in rev/s so to convert to rad/s: $(327.68)(2\pi) = 2058.87$ rad/s.

Exercise 2.3.3 (c): Calculate Max. Allowable Sampling Time (T_s)

For the experimental system, the max ω recorded was 477 rad/s (75.92rev/s). Our maximum pulse count (C_p) as calculated in part (b) is $32,768$. Using these figures along with our r value we can calculate the max allowable sampling time for our system (in seconds).

$$T_s = \frac{C_p}{r\omega} = \frac{(32768)}{(1000)(75.92)} = 0.4316$$

Section 3: Characterisation of 1st Order Systems

Exercise 3.5.1: Detailed Description of Experimental Process

In this experiment we will be analysing the response of a first order velocity control system to a step input [1]. We want to measure the response of the same two discs we used in Section 2 so $57g$ and $110g$. The procedure is as follows.

Starting with the $57g$ disc, we will use the "Voltage (open loop)" setting again but this time with a step input. We open the "graph" and "graph input" tools. We set the step size to $2V$ and press run. Depending on the disc, the response will take different amounts of time to settle at its final value. For $57g$ it takes about $7-10$ seconds. Save a CSV file of the response. Now that we've gathered the necessary data we can perform our analysis. We know that the voltage to speed transfer function of our 1st order system is given by:

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{K}{\tau s + 1} \tag{3.1}$$

Where $\Omega_m(s)$ is our angular velocity output, $V_m(s)$ is our input voltage, K is our system gain and τ is our time constant. We can calculate K and τ from our step response. Firstly, K is given by:

$$K = \frac{\Delta y}{\Delta u} = \frac{y_{ss} - y_0}{u_{max} - u_{min}} \tag{3.2}$$

where y_{ss} is our steady state final value, y_0 is our starting angular velocity and u_{max}/u_{min} are our maximum and minimum input signals. From figure 2 we can see that the steady state value of our $57g$ disc for a 2V step input is $\simeq 72$. Since both our input and output starts at 0 our value of K is given by: $K = \frac{72-0}{2-0} \simeq 36$.

Now we can calculate our time constant τ which is defined as the time taken for the system output to reach 63.2% $(1 - \frac{1}{e})$ of the final output. However, we must also subtract the time at which the step was applied (t_0). So if $y(t_1) = 0.632\Delta y + y_0$ then $\tau = t_1 - t_0$. Knowing this, and knowing that $\Delta y = 72$ then:

$$y(t_1) = 0.632\Delta y + y_0 = 0.632(72) + 0 = 45.504. \tag{3.3}$$

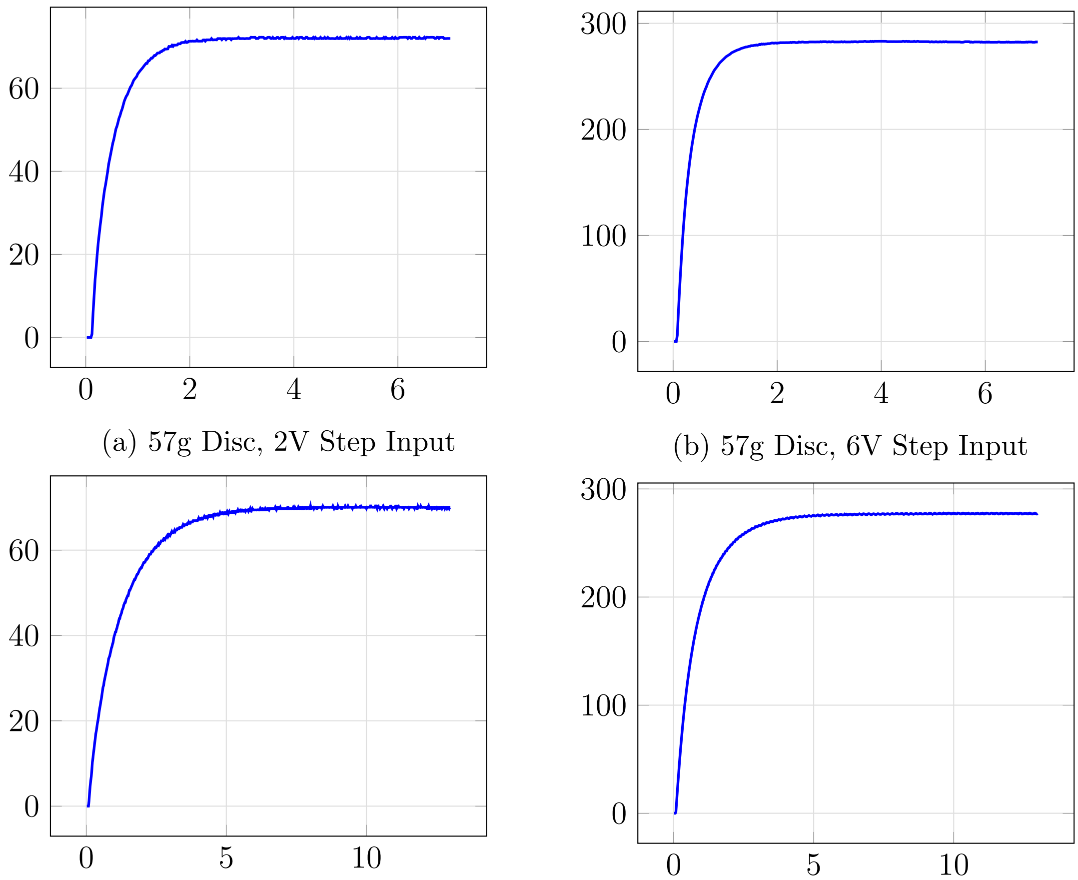
Inspecting our CSV table of date, we can see that the time at which our angular velocity is 45.504 is 0.5 seconds. Thus, $t_1 = 0.5$ seconds. We can also see from our CSV file that our step voltage was applied at 0.12 seconds so $t_0 = 0.12$. Therefore:

$$\tau = t_1 - t_0 = 0.5 - 0.12 = 0.38 \tag{3.4}$$

To verify our results we can use the inbuilt graphing tool on the remote lab interface. Selecting "Plot Function: Step" on the "graph" tool we can enter our calculations and plot a step response curve over our experimental curve. The curves match so our values are approximately accurate. We now repeat this experimental process for a $6V$ step input. We then repeat both step inputs for our second disc ($110g$). Note that for our $110g$ Disc experiments we need to allow a longer time to reach our steady state value. At each stage we download the CSV data file and perform the analysis as above. We can then produce a table of results as in table 1. We also plot the response curves for each separate experiment (figure 2).

Disk Info		2V Step			6V Step		
No.	Details	K	τ	t_0	K	τ	t_0
1	57g	36	0.38	0.12	47.1	0.265	0.1
2	110g	34.95	1.1	0.08	46.2	0.759	0.08

Table 1: Results of Step Response Experiments



(a) 57g Disc, 2V Step Input

(b) 57g Disc, 6V Step Input

(c) 110g Disc, 2V Step Input

(d) 110g Disc, 6V Step Input

Figure 2: Exercise 3.5.1, Step Response, Angular Velocity (ω) vs Time (s)

Exercise 3.5.2: Derive 1st Order Transfer Functions for each Disk

The transfer function of our 1st Order Velocity Control System is given by equation 3.1. Using values from table 1, the experimentally derived transfer functions are as follows. For the 57g Disc with a 2V step input the transfer function is:

$$\frac{\Omega_m(S)}{V_m(S)} = \frac{K}{\tau s + 1} = \frac{36}{0.38s + 1} \tag{3.5}$$

For the 57g Disc with a 6V step input the transfer function is:

$$\frac{\Omega_m(S)}{V_m(S)} = \frac{K}{\tau s + 1} = \frac{47.1}{0.265s + 1} \tag{3.6}$$

For the 110g Disc with a 2V step input the transfer function is:

$$\frac{\Omega_m(S)}{V_m(S)} = \frac{K}{\tau s + 1} = \frac{34.95}{1.1s + 1} \tag{3.7}$$

For the $110g$ Disc with a $6V$ step input the transfer function is:

$$\frac{\Omega_m(S)}{V_m(S)} = \frac{K}{\tau s + 1} = \frac{46.2}{0.759s + 1} \tag{3.8}$$

Exercise 3.5.3: Discuss and give reasons for the Discrepancy between 2V and 6V Transfer Functions

From figure 2 we can see the step responses with a $6V$ step input have steeper transient stages and therefore, they take a shorter time to reach their final steady state value. This is supported by our figures in table 1 as we can see that the time constants (τ) of the step responses with $6V$ step inputs are lower. For the $57g$ disc, the time constant is 43% higher for the $2V$ step. For the $110g$ disc, the time constant is 45% higher for the $2V$ step. In terms of our values for gain (K) the $6V$ step input produces a higher gain for both discs. The gain is 30% higher for the $57g$ disc and 32% higher for the $110g$ disc.

The reason for these discrepancies is likely to do with the torque of the system. As discussed in exercise 2.3.2 the torque of the system is the driving force that turns our disc load. Torque will be higher at a higher step voltage. Higher torque means higher acceleration according to the equation:

$$\frac{dw_m(t)}{dt} = \frac{\tau_m(t)}{J_{eq}} \tag{3.9}$$

where τ_m is our torque. The higher acceleration explains why the transient response for our $6V$ step input is so much faster (figure 2). The higher torque also causes the higher gain (K).

Exercise 3.5.4: Impact of Physical size of the Rotating Mass on the Transient and Steady State Response

The physical size of the disc affects its moment of inertia (J_d). The formula for moment of inertia of a body is given by:

$$J_d = \sum_{i=1}^{N} m_i r_i^2 \tag{3.10}$$

Where r_i is the distance the point masses lie from the axis of rotation. Our $110g$ disc has an outer radius $85mm$ while our $57g$ disc has an outer radius of $62mm$. Since more mass is distributed further out we can see from equation 3.10 that the moment of inertia of the bigger disc will be higher. A higher moment of inertia means the acceleration of our system will be lower according to equation 3.9 above, where J_{eq} is the total moment of inertia of the whole system. This will slow down our transient response and give us a higher value of τ. This is why our $110g$ disc has a slower transient response (see figure 2). Applying this theory to our design requirements for real world systems, if we want our radar surveillance system to have a fast transient response it makes sense to keep it as small and light as possible. Of course, this might be impractical or tricky to achieve.

Section 4: Validation using First Principles

Exercise 4.3.1: Derive Transfer Function of each Disc Tested

In this exercise we want to derive the transfer function of our system from first principles. Using the circuit diagram of our motor system and applying Kirchoff's Voltage Law, we get:

$$v_m(t) - R_m(t)i_m(t) - L_m\frac{di_m(t)}{dt} - e_b(t) = 0 \tag{4.1}$$

Where v_m is the input voltage, R_m is the motor resistance, i_m is the motor current, L_m is the motor inductance and e_b is the back EMF. The back EMF voltage is given by:

$$e_b(t) = K_m\omega_m(t) \tag{4.2}$$

where K_m is the motor constant and ω_m is our angular velocity. Since L_m is 0, combining equation 4.1 with 4.2 and rearranging for i_m we get:

$$i_m(t) = \frac{v_m(t) - K_m\omega_m(t)}{R_m} \tag{4.3}$$

Torque is related to current by the expression: $\tau_m(t) = K_t i_m(t)$, where K_t is the torque constant. Subbing equation 4.3 into this relationship we get:

$$\tau_m(t) = K_t(\frac{v_m(t) - K_m\omega_m(t)}{R_m}) \tag{4.4}$$

Torque is also given by the motor shaft equation: $\tau_m(t) = J_{eq}\frac{d\omega_m}{dt}$, where J_{eq} is the total moment of inertia of the system. Combining equation 4.4 with this expression gives:

$$J_{eq}\frac{d\omega_m(t)}{dt} = K_t(\frac{v_m(t) - K_m\omega_m(t)}{R_m}) \tag{4.5}$$

Now we perform a Laplace transform on both sides of our expression in order to convert to the s domain. This leaves us with:

$$J_{eq}s\Omega_m(s) = K_t(\frac{V_m(s) - K_m\Omega_m(s)}{R_m}) \tag{4.6}$$

Now we perform a series of rearrangements and then in order to get our transfer function into the same format as equation 3.1 we divide the top and bottom of the fraction by K_tK_m. This leaves us with:

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{K_t}{J_{eq}sR_m + K_tK_m} = \frac{\frac{K_t}{K_tK_m}}{\frac{J_{eq}R_m}{K_tK_m}s + 1} \tag{4.7}$$

For both discs $K_t = 0.02$Nm/A, $K_m = 0.02$Vs/rad and $R_m = 1.9\Omega$. For the $57g$ disc, the total moment of inertia J_{eq} of the system is $4.26 \times 10^{-5}kgm^2$. Therefore subbing these values into equation 4.7 the transfer function is given by:

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{\frac{0.02}{(0.02)(0.02)}}{\frac{(4.26\times10^{-5})(1.9)}{(0.02)(0.02)}s + 1} = \frac{50}{(0.20235)s + 1} \tag{4.8}$$

For the $110g$ disc, $J_{eq} = 11.8 \times 10^{-5}kgm^2$ so the final transfer function of the $110g$ disc is:

$$\frac{\Omega_m(s)}{V_m(s)} = \frac{\frac{0.02}{(0.02)(0.02)}}{\frac{(11.8\times10^{-5})(1.9)}{(0.02)(0.02)}s + 1} = \frac{50}{(0.5605)s + 1} \tag{4.9}$$

Exercise 4.3.2: Do these Transfer Functions match the ones in Section 3? If not why?

These transfer functions match the ones that we calculated experimentally in Section 3 quite well but they are not a perfect match. There are a few possible reasons for this. The first thing to notice is the transfer functions based on the $6V$ step in Section 3 are much closer approximations of the "theoretical" transfer functions than the $2V$ step transfer functions. However, even when compared to the transfer functions from the $6V$ step input we can see that there are discrepancies. From table 1, the $57g$ disc has a τ of 0.265 vs. the theoretical 0.20235 and the $110g$ disc has a τ of 0.759 vs 0.5605. The gain values (K) are also slightly lower.

The higher time constants of the experimentally derived transfer functions could be due to mechanical losses (wind/ air resistance). These factors would slow down our step response. Other inefficiencies in the DC motor such as magnetic losses could affect our τ and K values. Temperature fluctuations which are not present in theoretical systems could affect our step response. Finally, there could be non-conformity between the "designed" moment of inertia of the disc (J_{eq}) and the actual moment of inertia in the real world experiment which would affect the transfer function.

Exercise 4.3.3: Simulate in MATLAB the Derived Transfer Functions

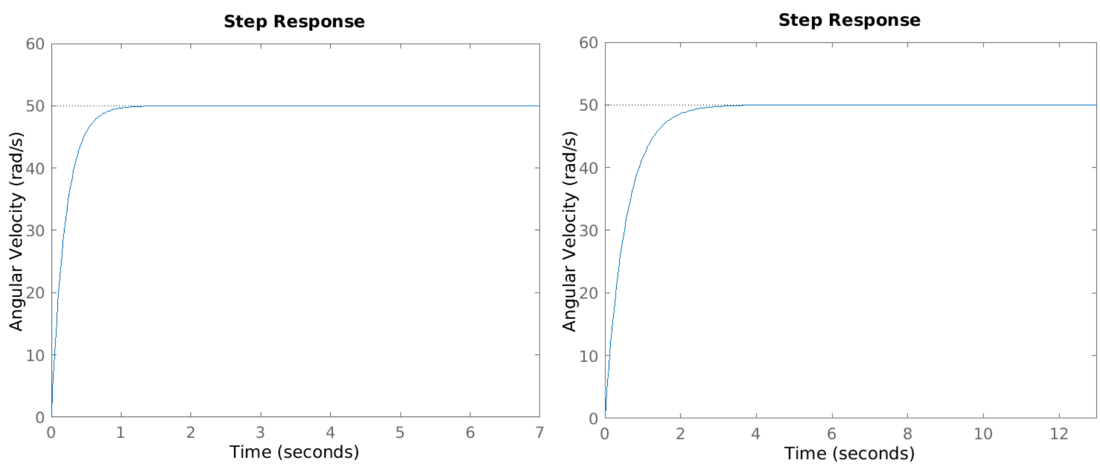
```
clearvars     % clear variables

tau = 0.20235;       % time constant value
K = 50;              % gain value
tFinal = 7;          % length of time of response

sys = tf(K,[tau, 1]);        % define transfer function

step(sys,tFinal)     % plot step response of transfer function
```

Using the `tf()` and `step()` MATLAB functions we produce the above code. Executing this script will plot our step response in a figure. We can edit the values of `tau`, `K` and `tFinal` to plot the theoretical transfer function of each disc (figure 3).



(a) 57g Disc Step Response (b) 110g Disc Step Response

Figure 3: MATLAB plots of theoretical step responses

Comparing these graphs to our experimental results, we can see the faster and steeper transient stage of the theoretical response, particularly when compared to the $2V$ responses. This was expected as we know the value of τ is lower. This is due to there being no real world influences. We also see there is no variation in the steady state region whereas in our experiments we saw minor fluctuations in the steady state value (y_{ss}).

Exercise 4.3.4: Equivalent RC Circuit

We must create a simple RC circuit that can mimic our system. We assume that the gain $(K) = 1$. An RC circuit is essentially a potential divider with a resistor (R) on top and a capacitor (with impedance X_c) on the bottom. The equation for this potential divider is:

$$V_{out} = V_{in} \times \frac{X_c}{R + X_c} \tag{4.10}$$

The impedance of a capacitor in the s (Laplace) domain is given by $X_c(s) = \frac{1}{sC}$. Therefore, subbing our impedance into the potential divider equation (equation 4.10) we get:

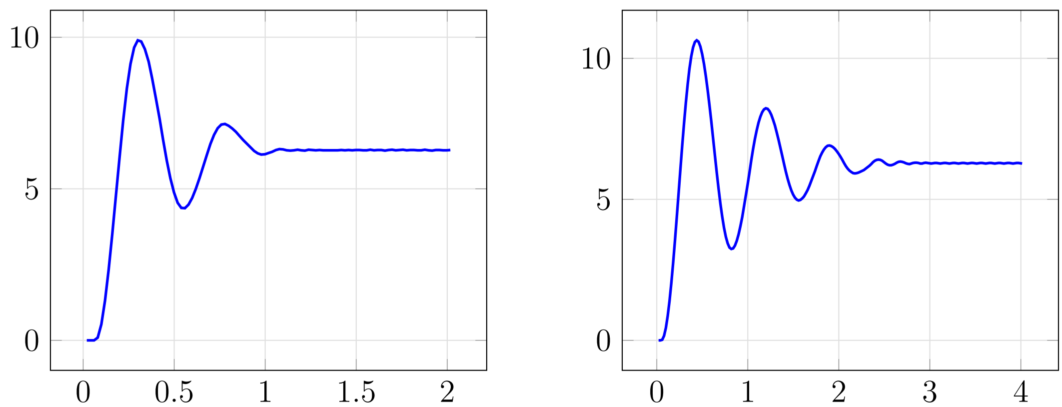
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs + 1} \tag{4.11}$$

Since we are assuming the gain of our theoretical system derived in Exercise 4.3.1 is 1 we can directly compare it to our RC transfer function (equation 4.11). Now, given that the capacitor size is $47\mu F$ we need to pick resistor values to replicate the *theoretical* transfer functions of our $57g$ and $110g$ systems. For our $57g$ disc system we know that $\tau = RC = 0.20235$. Therefore our calculation is given as: $R = \frac{\tau}{C} = \frac{0.20235}{47\times10^{-6}} = 4305.31\Omega$. Similarly, for our $110g$ disc system we know $\tau = RC = 0.5605$ therefore: $R = \frac{\tau}{C} = \frac{0.5605}{47\times10^{-6}} = 41925.53\Omega$

Section 5: Characterisation of 2nd Order Systems

Exercise 5.6.1: Calculate ω_n/ ζ and Derive Transfer Functions

In this exercise we must perform the experiment in Section 5 of the manual [1] and use our results to calculate the underdamped natural frequency (ω_n) and damping ratio (ζ) for each disc. Then we derive our transfer functions. Using the "Position (PID)" mode we record a second order response for both discs with a step input of 6.28 rads and save the data to a CSV file. For clarification the two responses are pictured in Figure 4 below.



(a) 57g Disc, 6.28 rads input

(b) 110g Disc, 6.28 rads input

Figure 4: Exercise 5.6.1, Second Order Step Response, Angular Position vs Time (s)

Firstly, we will perform the calculations for the 57g disc. The first step is to calculate the percentage overshoot (P.O) of our response. This is given by:

$$P.O = \frac{y_{max} - R_0}{R_0} \times 100 \tag{5.1}$$

Where y_{max} is the peak value of our response and R_0 is our input. For the 57g response given in figure 4 $P.O = \frac{9.9-6.28}{6.28} \times 100 = 57.64\%$. The next thing we calculate is t_p (the peak time) which is given by the time it takes the system to reach its max value (y_{max}) minus the time the step input is applied. For the 57g disc, $t_p = t_{max} - t_0 = 0.3 - 0.08 = 0.22$

We can now use our values for P.O and t_p to calculate the underdamped natural frequency and damping ratio for the 57g disc. Our damping ratio (ζ) and natural frequency (ω_n) are given by the following expressions. Subbing in our values for $P.O$ and t_p we get:

$$\zeta = \sqrt{\frac{\ln{(\frac{P.O}{100})^2}}{\pi^2 + \ln{(\frac{P.O}{100})^2}}} = \sqrt{\frac{\ln{(\frac{57.64}{100})^2}}{\pi^2 + \ln{(\frac{57.64}{100})^2}}} = 0.1727 \tag{5.2}$$

$$\omega_n = \frac{\pi}{t_p(\sqrt{1-\zeta^2})} = \frac{\pi}{(0.22)(\sqrt{1-(0.1727)^2})} = 14.4978 \tag{5.3}$$

We can now repeat these calculations for our 110g disc system. Doing so we will produce a table of results like table 2 below.

Disk Info		Measurements						Calculations	
No.	Details	R_0	y_{max}	t_0	t_{max}	t_p	P.O	ζ	ω_n
1	57g	6.28	9.9	0.08	0.3	0.22	57.64%	0.1727	14.4978
2	110g	6.28	10.64	0.06	0.439	0.379	69.43%	0.11536	8.3449

Table 2: Step Response Measurements for Second Order System

The final step of this exercise is to derive the second order transfer functions of our 57g and 110g systems. We use the standard transfer function of a second order dynamic system [1]. For our 57g disc system, subbing in our values from table 2, the transfer function equals:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{(14.4978)^2}{s^2 + 2(2.504)s + (14.4978)^2} = \frac{210.186}{s^2 + 5.008s + 210.186} \tag{5.4}$$

Similarly for our 110g disc system, subbing in our table 2 values the transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{(8.3449)^2}{s^2 + 2(0.96267)s + (8.3449)^2} = \frac{69.637}{s^2 + 1.9253s + 69.637} \tag{5.5}$$

Exercise 5.6.2: Using 1st Order Transfer Functions, Derive 2nd Order Transfer Functions

We know we can derive the angular position transfer function of our motor/disc assembly or "plant" ($P(s)$) by integrating the first order velocity transfer function (equation 3.1). Since in the s domain, integrating is the same as multiplying by $\frac{1}{s}$ we calculate:

$$P(s) = \frac{\Theta_m(s)}{V_m(s)} = \frac{K}{\tau s + 1} \times \frac{1}{s} = \frac{K}{s(\tau s + 1)} \tag{5.6}$$

Our overall second order system is a simple closed loop system composed of a compensator with transfer function $C(s) = 1$ and a motor/disc assembly with transfer function $P(s)$. Knowing this, and subbing in equation 5.6 for $P(s)$, we derive the overall transfer function of our second order system as:

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} \tag{5.7}$$

We can use this equation, along with our values of K and τ from Section 3 (table 1) to derive the transfer functions of our second order system for our $57g$ and $110g$ disc systems. There are 2 transfer functions for each disc ($2V$ and $6V$ step inputs). For the $57g$ disc (2V Step input), the transfer function is:

$$G(s) = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} = \frac{\frac{36}{0.38}}{s^2 + \frac{1}{0.38}s + \frac{36}{0.38}} = \frac{94.7368}{s^2 + 2.63s + 94.7368} \tag{5.8}$$

For the $57g$ disc system ($6V$ Step input), the transfer function is:

$$G(s) = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} = \frac{\frac{47.1}{0.265}}{s^2 + \frac{1}{0.265}s + \frac{47.1}{0.265}} = \frac{177.7258}{s^2 + 3.774s + 177.7358} \tag{5.9}$$

For the $110g$ disc system ($2V$ Step input), the transfer function is:

$$G(s) = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} = \frac{\frac{34.95}{1.1}}{s^2 + \frac{1}{1.1}s + \frac{34.95}{1.1}} = \frac{31.772}{s^2 + 0.909s + 31.772} \tag{5.10}$$

For the $110g$ disc system ($6V$ Step input), the transfer function is:

$$G(s) = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} = \frac{\frac{46.2}{0.759}}{s^2 + \frac{1}{0.759}s + \frac{46.2}{0.759}} = \frac{60.87}{s^2 + 1.317554s + 60.87} \tag{5.11}$$

Comparing these to our calculated second order transfer functions from Exercise 5.6.1 we see that the transfer functions based on the $6V$ step inputs are much better approximations than the $2V$ transfer functions.

Exercise 5.6.3: Calculate Peak Time (t_p) and P.O from the Transfer Functions in Exercise 5.6.2

We calculate the t_p and P.O of our transfer functions as follows. First for the transfer function of the $57g$ disc with $2V$ step input. Using the general transfer function of a second order system we know: $w_n = \sqrt{\frac{36}{0.38}} = 9.7333$ and $\zeta = \frac{1}{2\omega_n(0.38)} = 0.1352$. Therefore, according to the rearranged versions of equations 5.2 and 5.3:

$$P.O = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 100e^{-\frac{\pi(0.1352)}{\sqrt{1-(0.1352)^2}}} = 65.137\% \tag{5.12}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{9.7333(\sqrt{1-(0.1352)^2})} = 0.3258 \tag{5.13}$$

Now we can repeat for our other three transfer functions from Exercise 5.6.2. When we complete the calculations we can produce a table of results (table 3).

Disk Info		2V		6V	
No.	Details	t_p	P.O	t_p	P.O
1	57g	0.3258	65.137%	0.238	63.818%
2	110g	0.5592	77.566%	0.4041	76.62%

Table 3: Peak Time and P.O values of Exercise 5.6.2 Transfer Functions

Exercise 5.6.4: Discrepancies in Measured and Calculated Figures

Our calculated figures derived from our 6V step input are much closer to our Measured P.O and t_p from Exercise 5.6.1 than the calculated figures from our 2V step input which is to be expected (see exercise 3.5.3). However there are still minor discrepancies. For our $57g$ disc our t_p for our calculated values is 8.1% higher than our measured value and our P.O is 10% higher.

These discrepancies could come from the fact that our calculated values are based off our first order velocity system experiment from Section 3. Our disc encoder outputs *position data* so the data is being derived in this case to get velocity measurements. This process likely introduces some error. Our measured values are based on the direct position data outputted by the encoder during our second order system experiment.

Exercise 5.6.5: Steady State Error Calculations

First, lets consider our measured steady state errors. Using the "Position (PID)" setting we performed two ramp inputs: 1 rad/s for 10 seconds and 5 rad/s for 3 seconds, on each disc. The time for each ramp input was not long enough for the response to reach a steady state (oscillations were still present). So, in accordance with the lab manual, a linear fit was drawn from 1 second to near the end of the experiment. The steady state error was then measured as the vertical distance between these linear fits and our ramp input. The measured steady state errors are recorded in Table 4.

Now we can calculate our theoretical steady state errors using the final value theorem. According to the final value theorem the steady state error (e_{ss}) is given by:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s[R(s) - Y(s)] = \lim_{s\to 0} sR(s)[1 - \frac{Y(s)}{R(s)}] \tag{5.14}$$

So firstly for the $57g$ disc. For the step input steady state error, $R(s) = \frac{R_0}{s}$ where $R_0 = 6.28$, and $\frac{Y(s)}{R(s)}$ is the transfer function derived in Exercise 5.6.1. Therefore using equation 5.4,

$$e_{ss} = \lim_{s\to 0} s\frac{R_0}{s}[1 - \frac{210.186}{s^2 + 5s + 210.186}] = \lim_{s\to 0} 6.28[\frac{s^2 + 5s}{s^2 + 5s + 210.186}] = 0$$

Now for the ramp input, $R(s) = \frac{R_0}{s^2}$ where $R_0 = 1$ for the 1 rad/s ramp and $R_0 = 5$ for the 5 rad/s ramp. Therefore, for the 1 rad/s ramp, the ramp steady state error is:

$$e_{ss} = \lim_{s\to 0} s\frac{R_0}{s^2}[1 - \frac{210.186}{s^2 + 5s + 210.186}] = \lim_{s\to 0} \frac{1}{[}\frac{s+5}{s^2 + 5s + 210.186}] = \frac{5}{210.186} = 0.0238$$

And for the 5 rad/s ramp, the answer is simply $5[\frac{5}{210.186}] = 0.11894$ These calculations are repeated for the $110g$ disc and then the results are recorded in table 4. Inspecting our table of results we can see that our measured steady state errors are much higher than our theoretical errors. This is because our measured ramp inputs have only a finite time to settle while our theoretical ramp inputs have infinite time to settle.

Disk Info		Theoretical			Measured	
No.	Details	Step	Ramp		Ramp	
			1 rad/s	5 rad/s	1 rad/s	5 rad/s
1	57g	0	0.0238	0.11894	0.1792	0.252
2	110g	0	0.0276	0.138	0.0689	0.95544

Table 4: Steady State Error Results, Exercise 5.6.5

Section 6: PID Controller Design

Exercise 6.2.1: Assessment of the Stability of the System

For this section we will be using the $57g$ disc. Assuming a pure proportional (P) controller we must assess the stability of our system given by equation 5.7 where $C(s) = K_p$ instead of 1. So replacing $C(s)$ with K_p we calculate:

$$G(s) = \frac{K_p P(s)}{1 + K_p P(s)} = \frac{\frac{KK_p}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{KK_p}{\tau}} \tag{6.1}$$

Since we used the $57g$ disc in Section 5, and calculated values for ω_n and ζ (table 2), we know our value of $\tau = \frac{1}{2\zeta\omega_n} = 0.1997$ and $K = \omega_n^2\tau = 41.97$. Subbing in these values we get:

$$G(s) = \frac{\frac{(41.97)K_p}{0.1997}}{s^2 + \frac{1}{0.1997}s + \frac{(41.97)K_p}{0.1997}} = \frac{210.186K_p}{s^2 + 5.008s + 210.186K_p} \tag{6.2}$$

Now in order to assess the stability, we take the characteristic equation ($s^2 + 5.008s + 210.186K_p$) and do a simple routh table. We find:

s^2	1	$210.186K_p$
s^1	5.008	0
s^0	$210.186K_p$	0

The Routh-Hurwirz Criteria state that for the system to be stable NO sign change may occur, so $K_p > 0$. Therefore our system is stable for $\infty > K_p > 0$. The R-H criteria allow us to establish absolute stability, but do not tell us about relative stability.

Exercise 6.2.2: Select Appropriate Controller Configuration

The controller design chosen was a PD controller. The reasons are as follows. In exercise 5.6.5 we derived the theoretical steady state error of our system for a step input. We got a value of 0. The purpose of the Integral term is to remove any steady state error, but it also worsens our response time and stability. Since we have no steady state error there is no need to include the Integral term as it will worsen our response.

Including the Derivative term will improve the stability of our system which we assessed in Exercise 6.2.1. It will reduce overshoot and settling time which will give us a better response. Issues only arise using the K_d term if we are using it in isolation (we are using it with the Proportional term) or if there is substantial noise in the system, which there isn't.

Exercise 6.3.1: Describe the PID Controller Design Experiment

We are aiming to establish the parameters of our PD system. We select the $57g$ disc from the menu and use the "Position (PID)" setting. We select the "graph" option from the menu to display our response. Firstly, we have to record the uncontrolled system. Set $K_p = 1$, $K_i = 0$ and $K_d = 0$. Set the step input to 1 rad and run the experiment. Save the CSV of the data.

Now we will use the Z-N Ultimate gain method to calculate our parameters. Set our K_p to 4 and keep our step input at 1. Running the experiment we notice more oscillations. Keep running the experiment and adjusting K_p until we achieve a sustained oscillation, (our oscillations do not die out or grow). The value of K_p for which this is achieved is our *Ultimate Gain (K_u)*. For our experiment $K_u = 9.47$. Now we calculate our *Ultimate Period* (T_u) as the time one oscillation takes. A good way to calculate this is to measure the time it takes for 10 periods and divide by 10. Therefore: $T_u = \frac{1.597}{10} = 0.1597$ seconds.

Now using our values of K_u and T_u with the PD Z-N Table from Lecture 11 [2] we can find our parameters. So, for a PD system, $K_i = 0$ and:

$$K_p = 0.8 \times K_u = 0.8 \times 9.47 = 7.576 \tag{6.3}$$

$$K_d = T_d \times K_p = \frac{T_u}{8} \times K_p = \frac{0.597}{8} \times 7.576 = 0.15152 \tag{6.4}$$

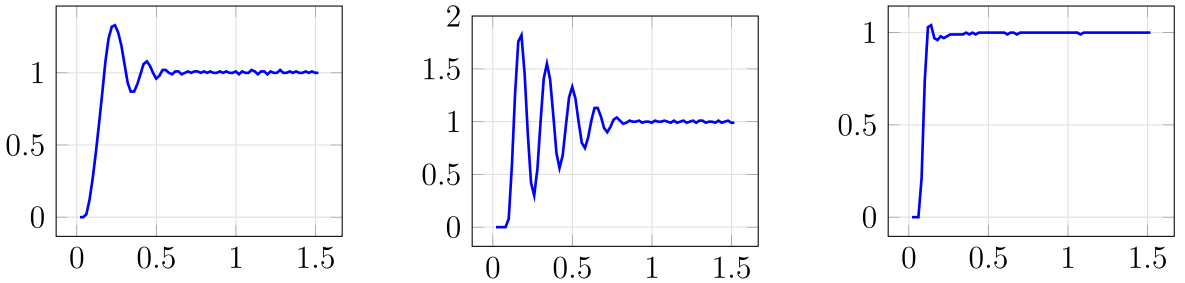
Plugging these parameters into our experiment and setting our step input to 1 rad again we run and record our response using Z-N derived parameters. Save the CSV of the data.

Finally we tweak our Z-N parameters to get an optimised response. We can use the guidelines on slide 11 of lecture 11 [2]. Normally we would adjust K_i first but since we are using a PD system we start with K_p. We want to achieve an overshoot of less than 10% and a rise time as fast as possible. According to slide 11, decreasing K_p will increase rise time and decrease overshoot so we decrease K_p from 7.576 to 5. We then reduce further from 5 to 3. Our overshoot is now about 50% but our settling time is still quite long.

We can now tweak our value of K_d. From slide 11 we know that increasing K_d will reduce our settling time and further reduce our overshoot. We increase K_d from 0.15152 to 0.5. This massively reduces our settling time and our overshoot is now just 28%. Increasing further to 1.5 gives us our desired overshoot of 4%. Our rise time is now very fast and our overall response follows our step input closely. Our steady state error, as predicted in Exercise 5.6.5 is 0. The parameters used for each response are included in table 5 and figures of all three responses are shown in figure 5.

Parameter	Uncontrolled	Z-N Parameters	Optimised Controller
K_p	1	7.576	3
K_i	0	0	0
K_d	0	0.15152	1.5

Table 5: PID Controller Parameters



(a) Uncontrolled (b) Z-N Parameters (c) Optimised Controller

Figure 5: PID Controller Experiment Responses

Exercise 6.3.2: Compare Experiment Results

Our Final optimised controller has a very fast rise time. The settling time is very short and the overshoot is very small. Our uncontrolled system actually has a better response than our Z-N system. This is not too unusual though as Z-N parameters often do not give a very optimal response. Instead, they serve as a good starting point to find our final optimised values. A table fully comparing the three responses is shown in table 6.

	P.O	Settling Time	Rise Time	SS Error
Uncontrolled	Poor P.O, 57.64%	2 oscillations before settling.	Slow Rise time, but stable	negligible
Z-N	Poor P.O, 82%	5 oscillations before settling.	Fast rise time, but unstable	negligible
Optimised	Very good P.O, Just 4%	Only one oscillation before settling.	Very fast rise time and stable	negligible

Table 6: Comparison of Responses

Exercise 6.3.3: Derive Full System Transfer Function

First, we consider the type of controller that was used. We used a PD controller so therefore $C(s) = K_p + K_d s$ since $K_i = 0$. We know from previous exercises that $P(s) = \frac{K}{s(\tau s+1)}$. Therefore using equation 5.7 we calculate our transfer function as:

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{(K_p + K_d s)(\frac{K}{s(\tau s+1)})}{1 + (K_p + K_d s)(\frac{K}{s(\tau s+1)})} \tag{6.5}$$

Now we perform some simplification and sub in our values: $K_p = 3$, $K_d = 5$ $K = 41.97$, $\tau = 0.1997$. We end up with the following equation, which is our Final Transfer Function:

$$G(s) = \frac{(\frac{K_d K}{\tau})s + (\frac{K_p K}{\tau})}{s^2 + (\frac{1+K_d K}{\tau})s + (\frac{K_p K}{\tau})} = \frac{(315.248)s + (630.4957)}{s^2 + (320.255)s + (630.4957)} \tag{6.6}$$

Exercise 6.3.4: Range of Values of K_i for a Stable System

We are trying to find the range of K_i for a stable system if the term was included in our controller. Firstly, we need to find our transfer function including our K_i parameter and then perform a Routh Table analysis. For a PID controller $C(s) = K_p + \frac{K_i}{s} + K_d$. Using the same derivation method used in Exercise 6.3.3 we find:

$$G(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{(K_p + \frac{K_i}{s} + K_d s)(\frac{K}{s(\tau s+1)})}{1 + (K_p + \frac{K_i}{s} + K_d s)(\frac{K}{s(\tau s+1)})} \tag{6.7}$$

$$G(s) = \frac{KK_d s^2 + KK_p s + K_i}{\tau s^3 + (1 + KK_d)s^2 + (KK_p)s + KK_i} \tag{6.8}$$

Taking our characteristic equation (the denominator) we perform a Routh Table analysis which yields:

s^3	τ	KK_p
s^2	$1 + KK_d$	KK_i
s^1	b_2	0
s^0	c_2	0

From this, according to the Routh-Hurwitz criteria, for the system to be stable $b_2 > 0$ and $c_2 > 0$. b_2 and c_2 are calculated by matrix algebra. Therefore:

$$b_2 = \frac{(1 + KK_d)(KK_p) - (\tau)(KK_i)}{1 + KK_d} > 0 \tag{6.9}$$

$$c_2 = \frac{(b_2)(KK_i) - (1 + KK_d)(0)}{b_2} = KK_i > 0 \tag{6.10}$$

From b_2, subbing in our values we get: $8052.574 - 0.1997K_i > 0$, $40323.86 > K_i$. And from c_2 we get: $K_i > 0$. Overall our range of values for K_i that give a stable system are:

$$40323.36 > K_i > 0 \tag{6.11}$$

Exercise 6.3.5: PID Design Compromises for Real World Systems

We learned when designing our PID controller that certain parameters have adverse affects as well as useful ones. For example, increasing our K_i term reduced steady state error to 0 but also negatively affected our stability and settling time. When making compromises in PID controller design its important to keep in mind the design requirements of the real world system. For example for a Tracking Radar system for aeroplanes, we require our response to be very fast and immune to disturbances. This means we require a high K_d term to maximise our stability and minimise settling time. Other systems may not require as fast a response but they need very finely tuned accuracy like a temperature sensor. This means we can't afford any steady state error so we might have a value for our K_i term at the cost of stability and rise time.

We must also consider the fact that our parameters K_p, K_i and K_d all affect each other, so changing one parameter can have an effect on the others. Therefore, we must perform tests and calibrations after changing our parameters to ensure the changes have given us the desired response.

Bibliography

[1] Dr. Aristides E. Kiprakis. *Control and Instrumenation Engineer 3 Laboratory Workbook.* (accessed: 22.04.2021).

[2] Dr. Aristides E. Kiprakis. *Lecture 11: PID Controllers.* (accessed: 22.04.2021).